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A Compact Dissipative Dynamical System for a Difference Equation with Diffusion

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1. INTRODUCTION

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} + f(u) & (t, x) \in (0, \infty) \times [0, 1] \\ u(t, 0) = u(t, 1) = 0 & t > 0 \end{cases} \tag{1}$$

was treated in [1], taking a sequence $\{u_n\}_{n \in \mathbb{N}}$ of functions $u_n \in H_0^1 = \{p: [0, 1] \to \mathbb{R};\ p$ is absolutely continuous with $p(0) = p(1) = 0$ and $p' \in L_2\{0, 1)\}$ satisfying:

$$u_{n+1} - u_n = \varepsilon\, \Delta u_{n+1} + \varepsilon f \circ u_n \qquad n \in \mathbb{N}, \quad \Delta = \frac{\partial^2}{\partial x^2}, \text{ for a given } \varepsilon > 0. \tag{2}$$

In that paper, the authors studied iteration of the map $\Phi_\varepsilon: H_0^1 \to H_0^1$, $\Phi_\varepsilon = (I - \varepsilon \Delta)^{-1} \circ (I + \varepsilon \hat{f})$ (where $\hat{f}(p) = f \circ p$ for $p \in H_0^1$ and I is the identity of H_0^1), since $\{u_n\}_{n \in \mathbb{N}}$ satisfies (2) iff $u_{n+1} = \Phi_\varepsilon(u_n)$, $n \in \mathbb{N}$.

We will consider here the version of (2) with continuous argument, given by:

$$\begin{cases} u(t, x) - u(t - r, x) = \varepsilon \dfrac{\partial^2 u}{\partial x^2}(t, x) + \varepsilon f(u(t - r, x)) & (t, x) \in (0, \infty) \times [0, 1] \\ u(t, 0) = u(t, 1) = 0 & t > 0 \end{cases} \tag{3}$$

with r, ε positive constants, f of class C^1 with $\sup_{y \in \mathbb{R}} |f'(y)| = K < \infty$ and $f(0) = 0$.

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It seems reasonable that (3) is an approximation of (1) when $r=\varepsilon\to 0$.

We will study the dynamical system defined by (3) in the Hilbert space $L_2^\# \stackrel{\text{def}}{=} L_2([-r,0], H_0^1)$. We show (3) defines a nonlinear semigroup of operators from $L_2^\#$ to itself, $\{S(t), t\geqslant 0\}$, strongly-continuous in time. Taking f of class C^2 with bounded second derivative, we show that, for each $t>0$, $S(t)\colon L_2^\# \to L_2^\#$ is Lipschitzian and has a strongly-continuous Hadamard derivative, though it is nowhere Frechet differentiable. For dissipativity, we assume also $\varepsilon K<1$ and $\lim_{|y|\to\infty} \sup f(y)/y<\pi^2$. Then we find a closed bounded attractor $\mathscr{A}\subset L_2^\#$, which attracts every compact set. In case $K\leqslant\pi^2$ we have $\mathscr{A}=\{0\}$ and it attracts all bounded sets of $L_2^\#$, but if there is a non-zero equilibrium (for example, if $f'(0)>\pi^2$), $\mathscr{A}$ is not compact. There are finitely many equilibria and in general (for most choices of f) they are all hyperbolic: the linearization of the nonlinear semigroup is a C^0-linear semigroup with a hyperbolic splitting (and, typically, infinite-dimensional unstable space). But when there is some non-zero equilibrium, the equilibria are *not* isolated invariant sets: every neighborhood of each equilibrium contains non-constant periodic orbits (with period r). Indeed, every neighborhood contains uncountably many hyperbolic r-periodic orbits: fixed points of $S(r)$ such that the derivative has spectrum disjoint from the unit circle.

This radical departure from the expected behavior near a "hyperbolic" element is presumably due to the use of a Hadamard derivative, rather than a Frechet derivative. Perhaps the infinite-dimensional unstable space plays a role. In any case, there is no hope for a Hartman–Grobman result—the author's initial motivation. We note that Kening Lu, in [7], proved the Hartman–Grobman result for system (1) in H_0^1.

We recall the definition of Hadamard derivative for an operator $S\colon E\to F$, with E, F Banach spaces. We say that S is Hadamard-differentiable at $q\in E$ if there exists a linear map $S'(q)\colon E\to F$ such that for each $h\in E$ we can write $S(q+tk+th)=S(q)+tS'(q)\,h+tR(t,k,h)$ where $R(t,k,h)\to 0$ as $(t,k)\to(0,0)$ in $\mathbb{R}\times E$. In this case the (unique) map $S'(q)$ is then called the Hadamard derivative of S at q (see [2] for details).

2. THE FUNDAMENTAL THEORY

We consider in H_0^1 the norm $\|\ \|_{H_0^1}$ coming from the inner product $\langle p,q\rangle=\int_0^1 p'(x)\cdot q'(x)\,dx$. We remark that for p in H_0^1 we have $\|p\|_2\leqslant \|p\|_\infty\leqslant\|p\|_{H_0^1}$. Moreover $\pi\|p\|_2\leqslant\|p\|_{H_0^1}$. We put $L_2=L_2(0,1)$, and remark that, when f is C^1, $f(0)=0$ and $\sup_{y\in\mathbb{R}}|f'(y)|=K<\infty$, we have

$$\hat{f}(p)=f\circ p\in H_0^1$$

and

$$\|\hat{f}(p)\|_{H_0^1} = \left(\int_0^1 [f'(p(x)) \cdot p'(x)]^2 \, dx \right)^{1/2} \leqslant K \|p\|_{H_0^1}$$

and

$$\|\hat{f}(p) - \hat{f}(q)\|_2 = \left(\int_0^1 (f(p(x)) - f(q(x)))^2 \, dx \right)^{1/2}$$

$$\leqslant K \|p - q\|_2 \leqslant K \|p - q\|_{H_0^1}, \qquad \forall p, q \in H_0^1.$$

Then, using the above observations and the fact that f' is uniformly continuous in each bounded interval of $\mathbb{R}$ we get the

PROPOSITION 2.1. *$\hat{f}: H_0^1 \to L_2$ is of class C^1 and Lipschitzian and $\hat{f}(H_0^1) \subset H_0^1$.*

We note that $\hat{f}': H_0^1 \to \mathscr{L}(H_0^1, L_2)$ is such that $[\hat{f}'(p) \cdot \Delta p](x) = f'(p(x)) \, \Delta p(x)$ for all p, $\Delta p \in H_0^1$ and $x \in [0, 1]$.

The system (3) is equivalent to

$$U(t) - U(t-r) = \varepsilon \, \Delta U(t) + \varepsilon \hat{f}(U(t-r)) \qquad t > 0, \quad \text{in } H_0^1 \tag{4}$$

and (4) is equivalent to the difference equation in H_0^1:

$$U(t) = \Phi(U(t-r)) \qquad t > 0. \tag{5}$$

with $\Phi = \Phi_\varepsilon$ as in the introduction, $\Phi = Q \circ (I + \varepsilon \hat{f})$, $Q = (I - \varepsilon \Delta)^{-1}$. It is easy to see that $Q \in \mathscr{L}(H_0^1)$, $Q \in \mathscr{L}(L_2)$, and in both cases we have $\|Q\| = 1/1 + \varepsilon\pi^2$ (it suffices to take the orthonormal basis $\{s_n\}_{n \in \mathbb{N}^*}$, $s_n = \sin(n\pi \cdot)/n\pi$ for the Hilbert space H_0^1, or $s_n = \sin(n\pi \cdot)$ for L_2, and verify that for $p = \sum p_n s_n$ we have $Qp = \sum (p_n s_n)/(1 + \varepsilon\pi^2 n^2)$ in both cases).

It is easy to verify also that $Q_{|H_0^1} \in \mathscr{L}(L_2, H_0^1)$. Let $\|Q\|^*$ be its norm in $\mathscr{L}(L_2, H_0^1)$; $\|Q\|^* \leqslant 1/\varepsilon\pi$.

PROPOSITION 2.2. *$\Phi: H_0^1 \to H_0^1$ is of class C^1 and Lipschitzian and we have*:

$$\|\Phi(p)\|_{H_0^1} \leqslant \frac{1 + \varepsilon K}{1 + \varepsilon\pi^2} \cdot \|p\|_{H_0^1}$$

and

$$\|D\Phi(p)\| \leqslant (1 + \varepsilon K) \, \|Q\|^* \quad \forall p \in H_0^1.$$

Proof. We get the first inequality above when we see $\Phi: H_0^1 \xrightarrow{I+\varepsilon\hat{f}} H_0^1 \xrightarrow{Q} H_0^1$ but for the rest of the proposition we consider $\Phi: H_0^1 \xrightarrow{I+\varepsilon\hat{f}} L_2 \xrightarrow{Q} H_0^1$ and use Proposition 2.1. ▌

In these conditions we may see equation (5) in the phase space $L_2^\# \stackrel{\text{def}}{=} L_2([-r, 0], H_0^1)$ as a particular case of the neutral equation studied in [2], more precisely, we take equation (3.2) of [2] and put $\xi = 0$, $F \equiv 0$ and $\mathscr{D}(U_t) = E(U(t), U(t-r))$ where $E(p_o, p_r) = p_o - \Phi(p_r)$, for p_o, p_r in H_0^1. We remind that, for the neutral equations, given a function U and a real number t, we define $U_t(\theta) = U(t+\theta)$ for θ in $[-r, 0]$, whenever the interval $[t-r, t]$ is contained in the domain of U.

Then, Theorem 3.1 of [2] applies to our case giving the

THEOREM 2.1. *For each $\varphi \in L_2^\#$, there exists an unique solution $U \in L_2^{loc}([-r, \infty), H_0^1)$ of equation (5) such that $U_0 = \varphi$. Moreover, the application $(t, \varphi) \in [0, \infty) \times L_2^\# \to U_t \in L_2^\#$ is continuous.*

Thus, we obtain the flow $\{S(t)\}_{t \geqslant 0}$ of equation (5), C^0-semigroup of continuous operators $S(t): L_2^\# \to L_2^\#$, $S(t)\,\varphi = U_t$, with U given in Theorem 2.1.

It is easy to see that if f is of class C^2 with $\sup_{y \in \mathbb{R}} |f''(y)| \leqslant \bar{K} < \infty$, then, using the uniform continuity of f'' in bounded intervals of $\mathbb{R}$, we obtain that $\hat{f}: H_0^1 \to L_2$ is of class C^2 with $\sup_{p \in H_0^1} \|\hat{f}''(p)\| \leqslant \bar{K}$ and $\Phi: H_0^1 \to H_0^1$ is of class C^2 with $\sup_{p \in H_0^1} \|D^2\Phi(p)\| \leqslant \varepsilon \|Q\|^* \cdot \bar{K}$.

Then, we may adapt Theorem 3.2 of [2] to our case and obtain the:

THEOREM 2.2. *If f is of class C^2 with f'' bounded on $\mathbb{R}$ then, for all $t \geqslant 0$, $S(t): L_2^\# \to L_2^\#$ is strongly-continuous-Hadamard-differentiable at any $\varphi \in L_2^\#$ and, for $\Delta\varphi \in L_2^\#$ we have $(DS(t)\,\varphi)\,\Delta\varphi = Z_t$ where Z is the solution of the linearized equation around the solution $U_t = S(t)\,\varphi$ given by:*

$$Z(t) = D\Phi(U(t-r)) \cdot Z(t-r) \qquad \text{for} \quad t > 0 \tag{6}$$

with $Z_0 = \Delta\varphi$.

3. DISSIPATIVITY

Here we add the hypothesis that $\varepsilon K < 1$ which is used in Lemma 2.1 of [1].

In [1] there is defined a Liapunoff function $V: H_0^1 \to \mathbb{R}$, for the dynamic of Φ, by

$$V(p) = \int_0^1 \left[\tfrac{1}{2}\, p'(x)^2 - F(p(x))\right] dx$$

for p in H_0^1, where

$$F(y) = \int_0^y f(s)\, ds, \qquad y \in \mathbb{R}.$$

By Lemma 2.1 of [1] we have $V(\Phi(p)) \leqslant V(p)$, $\forall p \in H_0^1$ and the equality occurs iff $\Phi(p) = p$.

From now on we will admit a dissipative condition for f:

$$\overline{\lim_{|y| \to \infty}} \frac{f(y)}{y} < \pi^2 \qquad \left(\text{that is, } \exists M > 0 \text{ and } 0 < k < 1 \mid |y| \geqslant M \Rightarrow \frac{f(y)}{y} \leqslant k\pi^2\right) \tag{7}$$

with this condition satisfied we get the:

PROPOSITION 3.1.

$$\frac{1-k}{2} \|p\|_{H_0^1}^2 - \frac{KM^2}{2} \leqslant V(p) \leqslant \frac{1+K}{2} \|p\|_{H_0^1}^2 \qquad \forall p \in H_0^1.$$

Proof. For $y \in \mathbb{R}$,

$$F(y) = \int_0^1 f(\xi y)\, y\, d\xi = \int_0^1 f'(\eta_{y,\xi})\, \xi y y\, d\xi,$$

so

$$|F(y)| \leqslant \tfrac{1}{2} K y^2.$$

If $|y| \leqslant M$ we have

$$F(y) \leqslant \tfrac{1}{2} KM^2.$$

If $|y| \geqslant M$,

$$\frac{1}{y} \frac{d}{dy} \left(F(y) - \frac{k\pi^2}{2} y^2\right) = \frac{f(y)}{y} - k\pi^2 \leqslant 0$$

and then

$$F(y) \leqslant \frac{k\pi^2}{2} y^2.$$

So

$$-\frac{1}{2}Ky^2 \leqslant F(y) \leqslant \frac{k\pi^2 y^2}{2} + \frac{1}{2}KM^2 \qquad \text{for all} \quad y \in \mathbb{R}.$$

Taking $y = p(x)$ and integrating from $x = 0$ to $x = 1$ we obtain

$$-\frac{1}{2}K\|p\|_2^2 \leqslant \int_0^1 F(p(x))dx \leqslant \frac{k\pi^2}{2}\|p\|_2^2 + \frac{1}{2}KM^2 \leqslant \frac{k}{2}\|p\|_{H_0^1}^2 + \frac{KM^2}{2}$$

and from this the proposition follows. ∎

Definition 3.1. Let us define

$$\mathscr{V} : L_2^{\#} \to \mathbb{R}$$

by

$$\mathscr{V}(\varphi) = \int_{-r}^{0} V(\varphi(\theta))\, d\theta \qquad \forall \varphi \in L_2^{\#}.$$

We note that in the orbits of (5) we have

$$\mathscr{V}(U_t) = \int_{t-r}^{t} V(U(\sigma))\, d\sigma$$

and then

$$\begin{aligned} \frac{d}{dt}\mathscr{V}(U_t) &= V(U(t)) - V(U(t-r)) \\ &= V(\Phi(U(t-r))) - V(U(t-r)) \end{aligned}$$

and by Lemma 2.1 of [1], mentioned above, we conclude that

$$\frac{d}{dt}\mathscr{V}(U_t) \leqslant 0$$

and the equality occurs if and only if $U(t) = U(t-r)$.

Let E be the set of fixed point of Φ in H_0^1, that is, $E = \{p \in H_0^1 \mid \Phi(p) = p\}$ and put $\mathscr{E} = \{\varphi \in L_2^{\#} \mid \varphi(\theta) \in E$ a.e. for θ in $[-r, 0]\}$.

Proposition 3.2. *$\mathscr{V}$ is continuous, bounded below, $\mathscr{V}(\varphi) \to \infty$ when $\|\varphi\|_2^{\#} \to \infty$, $\mathscr{V}$ is bounded in bounded sets of $L_2^{\#}$ and $\mathscr{V}(S(t)\varphi)$ is non-increasing in t for each φ in $L_2^{\#}$. Moreover, if φ is such that $S(t)\varphi$ is defined for all t in $\mathbb{R}$ and $\mathscr{V}(S(t)\varphi) = \mathscr{V}(\varphi)$, $\forall t \in \mathbb{R}$, then $\varphi \in \mathscr{E}$.*

Proof. For the continuity of $\mathscr{V}$ in $\varphi \in L_2^{\#}$, take $\bar{\varphi}$ near of φ and let c be a positive constant such that $\|\varphi\|_2^{\#} < c$, $\|\bar{\varphi}\|_2^{\#} < c$. We have:

$$\begin{aligned}
&|\mathscr{V}(\varphi) - \mathscr{V}(\bar{\varphi})| \\
&\quad \leqslant \tfrac{1}{2} \big| \, \|\varphi\|_2^{\#^2} - \|\bar{\varphi}\|_2^{\#^2} \big| + \int_{-r}^{0} \int_0^1 |F(\varphi(\theta, x)) - F(\bar{\varphi}(\theta, x))| \, dx \, d\theta \\
&\quad \leqslant c \, \|\varphi - \bar{\varphi}\|_2^{\#} + \int_{-r}^{0} \int_0^1 K \, |\alpha(\theta, x)| \cdot |\varphi(\theta, x) - \bar{\varphi}(\theta, x)| \, dx \, d\theta
\end{aligned}$$

where $\alpha(\theta, x)$ is between $\varphi(\theta, x)$, and $\bar{\varphi}(\theta, x)$, thus $|\alpha(\theta, x)| \leqslant |\varphi(\theta, x)| + |\bar{\varphi}(\theta, x)|$, then:

$$\begin{aligned}
&|\mathscr{V}(\varphi) - \mathscr{V}(\bar{\varphi})| \\
&\quad \leqslant c \, \|\varphi - \bar{\varphi}\|_2^{\#} + K \int_{-r}^{0} (\|\varphi(\theta, \cdot)\|_2 + \|\bar{\varphi}(\theta, \cdot)\|_2) \cdot \|\varphi(\theta, \cdot) - \bar{\varphi}(\theta, \cdot)\|_2 \, d\theta \\
&\quad \leqslant c \, \|\varphi - \bar{\varphi}\|_2^{\#} + K \, 2c \, \|\varphi - \bar{\varphi}\|_2^{\#},
\end{aligned}$$

then $\mathscr{V}$ is continuous in φ.

Integrating from $-r$ to 0 the expression of Proposition 3.1, applied in $\varphi(\theta)$, we get:

$$\frac{1-k}{2} \, \|\varphi\|_2^{\#^2} - \frac{KM^2 r}{2} \leqslant \mathscr{V}(\varphi) \leqslant \frac{1+K}{2} \, \|\varphi\|_2^{\#^2}$$

and using this and the observation done in Definition 3.1 we conclude the proposition. ∎

We see, for p in H_0^1, that p is in E if and only if p satisfies

$$\begin{cases} p''(x) + f(p(x)) = 0, & 0 \leqslant x \leqslant 1 \\ p(0) = p(1) = 0. \end{cases} \tag{8}$$

The solution p is "simple" if zero is not an eigenvalue of the linearization (i.e., if there is no non-trivial fixed point of $D\Phi(p)$), and this holds for most choices of f. For example, given any f_0 of class C^1, if $f = \lambda f_0$ with $\lambda \in \mathbb{R}$; for a dense (residual) set of $\lambda \in \mathbb{R}$, all solutions p are simple, according to [5], Theorem 11. This is equivalent to say $1 \notin \sigma(D\Phi(p))$ for all $p \in E$, where $\sigma(D\Phi(p))$ is the spectrum of $D\Phi(p)$.

Put $L_\infty^{\#} = L_\infty([-r, 0], H_0^1)$. We have the:

PROPOSITION 3.3. *E is bounded in H_0^1 and $\mathscr{E}$ is bounded in $L_2^{\#}$. Moreover, $\mathscr{E} \subset L_\infty^{\#}$ and $\mathscr{E}$ is bounded in $L_\infty^{\#}$ (with the $\| \; \|_\infty^{\#}$-norm).*

Proof. Let p be in E, then $p'' \cdot p + f(p)\,p = 0 \Rightarrow \int_0^1 p''p\,dx = -\int_0^1 pf(p)\,dx$ and integrating by parts the first member we have:

$$-\int_0^1 p'^2\,dx = -\int_0^1 pf(p)\,dx,$$

then

$$\|p\|^2_{H_0^1} = \int_{|p(x)| < M} pf(p)\,dx + \int_{|p(x)| > M} pf(p)\,dx$$

so

$$\|p\|^2_{H_0^1} \leqslant KM^2 + k\pi^2\,\|p\|_2^2 \leqslant KM^2 + k\,\|p\|^2_{H_0^1}$$

(with M and k given in condition (7)), i.e.,

$$\|p\|^2_{H_0^1} \leqslant \frac{KM^2}{1-k} = d.$$

Then, if $\varphi \in \mathscr{E}$ we will have $\|\varphi\|_\infty^{\#^2} \leqslant d$ and $\|\varphi\|_2^{\#^2} \leqslant dr$. ∎

Remark 3.1. The elements of $\mathscr{E}$ are the fixed points of $\{S(t)\}_{t \geqslant 0}$ and the periodic points which have the period r or divisor of r, i.e., $\varphi \in \mathscr{E} \Leftrightarrow S(r)\,\varphi = \varphi$. We verify also that the flow of (5) does not have fixed or periodic points outside of $\mathscr{E}$, since $\mathscr{V}(S(t+r)\,\varphi) < \mathscr{V}(S(t)\,\varphi)$, when $S(t)\,\varphi \notin \mathscr{E}$.

Remark 3.2. $0 \in E \subset H_0^1$, because $f(0) = 0$. If $E = \{0\}$ then $\mathscr{E} = \{0\} \subset L_2^\#$.

If E has some element $v \neq 0$ then $\mathscr{E}$ is not precompact since the set $\{v\mathscr{X}_G \mid G \subset [-r, 0]$ is Lebesgue-measurable$\} \subset \mathscr{E}$ is not precompact in $L_2^\#$. For example if

$$G_n = \bigcup_{j=0}^{n-1} \left[\frac{-r}{n}\left(j+\frac{1}{2}\right), \frac{-r}{n}\,j\right],$$

$v\mathscr{X}_{G_n}$ tends weakly to the constant $\frac{1}{2}v$ in $L_2^\#$ but

$$\left\|v\mathscr{X}_{G_n} - \frac{1}{2}\,v\right\|_2^\# = \frac{\sqrt{r}}{2}\,\|v\|_{H_0^1}.$$

(We define the function $v\mathscr{X}_G$ as $v\mathscr{X}_G(\theta) = v$ when $\theta \in G$ and $v\mathscr{X}_G(\theta) = 0$ otherwise).

PROPOSITION 3.4. *If* $\sup_{y \in \mathbb{R}} |f'(y)| = K < \pi^2$ *then* $\mathscr{E} = \{0\} \subset L_2^{\#}$ *and is global attractor for the flow of* (5).

Proof. Let φ be in $L_2^{\#}$ and U be the solution of (5) with $U_0 = \varphi$, that is, $S(t)\,\varphi = U_t$ for $t \geqslant 0$. We have

$$\begin{aligned}
\|S(r)\,\varphi\|_2^{\#} &= \|U_r\|_2^{\#} \\
&= \left(\int_{-r}^{0} \|U(r+\theta)\|_{H_0^1}^2 \,d\theta\right)^{1/2} \\
&= \left(\int_{-t}^{0} \|\Phi(\varphi(\theta))\|_{H_0^1}^2 \,d\theta\right)^{1/2} \\
&\leqslant \left(\int_{-r}^{0} \left[\frac{1+\varepsilon K}{1+\varepsilon\pi^2} \|\varphi(\theta)\|_{H_0^1}\right]^2 d\theta\right)^{1/2} \\
&= \frac{1+\varepsilon K}{1+\varepsilon\pi^2} \|\varphi\|_2^{\#}
\end{aligned}$$

(using Proposition 2.2) and

$$\frac{1+\varepsilon K}{1+\varepsilon\pi^2} = c < 1.$$

Then, by induction, $\|S(nr)\,\varphi\|_2^{\#} = \|U_{nr}\|_2^{\#} \leqslant c^n \|\varphi\|_2^{\#}$ for $n \in \mathbb{N}$, and we find positive constants a and L such that $\|S(t)\,\varphi\|_2^{\#} \leqslant Le^{-at} \|\varphi\|_2^{\#}$, $\forall t \geqslant 0$. Then, given a bounded set $B \subset L_2^{\#}$ ($\exists b > 0$ such that $\varphi \in B \Rightarrow \|\varphi\|_2^{\#} \leqslant b$), for each $\delta > 0$ we find $\sigma > 0$ such that for $t \geqslant \sigma$ we have $S(t)\,B \subset \mathscr{B}_\delta(0)$ (ball of $L_2^{\#}$). This is true because for $\varphi \in B$ we have: $\|S(t)\,\varphi\|_2^{\#} \leqslant Le^{-at} \|\varphi\|_2^{\#} \leqslant Le^{-a\sigma} b < \delta$. ∎

Remark 3.3. Results obtained in [1]:

The dynamical system of Φ in H_0^1 has a global compact attractor $A \subset H_0^1$ which is the union of all global and bounded orbits of Φ. The set of non-wandering points of Φ reduces to E and attracts points of H_0^1 (i.e., the ω-limit set of p, $\omega(p)$, is in E, for all p in H_0^1). For all $v \in E$, the linear operator $D\Phi(v)$ is compact and self-adjoint in an equivalent norm of H_0^1

$$\left(\|v\| = \left[\int_0^1 \left(v'^2(x) + \frac{v^2(x)}{\varepsilon}\right) dx\right]^{1/2}\right).$$

The eigenvalues of $D\Phi(v)$ are all simple and positive, when $\varepsilon K < 1$, and can be ordered as $\lambda_0 > \lambda_1 > \lambda_2 \cdots > \lambda_n$, with $\lambda_n \to 0$ when $n \to \infty$. If $1 \notin \sigma(D\Phi(v))$, which is true for most choices of f as noted before, v is a hyperbolic fixed point of Φ.

If you suppose that all fixed points of Φ are hyperbolic it follows that E is finite, $E=\{0, v_1, v_2, ..., v_m\}$. Then $A=W^u(E)$, the unstable manifold of E, and Φ will be a Morse–Smale map.

Remark 3.4. Using Lemma 3.8.4 of [3] we see that if all fixed points of Φ are hyperbolic then, for each p in H_0^1, there is a v in E such that $\omega(p)=\{v\}$ and then $\lim_{n\to\infty} \Phi^n(p)=v$. Thus,

$$H_0^1=W^s(E)=\bigcup_{i=0}^{m} W^s(v_i) \qquad \text{(stable manifolds).}$$

Using the observation after Theorem 2.7.1 of [3] we conclude that

$$A=W^u(E)=\bigcup_{i=0}^{m} W^u(v_i).$$

From now on we suppose that all fixed points of Φ are hyperbolic, $E=\{0, v_1, ..., v_m\}$. Then $\mathscr{E}=\{\sum_{i=0}^m v_i \mathscr{X}_{G_i} \mid \{G_i\}_{i=0,...,m}$ is a Lebesgue-measurable disjoint covering of $[-r, 0]\}$ is the set of all simple functions of $L_2^\#$ with values in E.

For each $\varphi\in L_2^\#$ we can define $\psi(\theta)=\lim_{n\to\infty} \Phi^n(\varphi(\theta))$ a.e. for θ in $[-r, 0]$ that exists by Remark 3.4. ψ is measurable (a pointwise limit of measurable functions) and $\psi(\theta)\in E$ a.e. in θ. Hence $\psi\in\mathscr{E}$.

THEOREM 3.1. *For $\varphi\in L_2^\#$, take $\psi\in\mathscr{E}$ as above, i.e., $\psi(\theta)=\lim_{n\to\infty} \Phi^n(\varphi(\theta))$ a.e. in θ. Then $\gamma(\psi)$ attracts φ (where $\gamma(\psi)=\{S(t)\,\psi \mid t\in\mathbb{R}\}$ is the orbit of ψ).*

Proof. Let U be the solution of (5) by φ, i.e., $S(t)\,\varphi=U_t$ for $t\geqslant 0$. Let $g_n\in L_2([-r, r], H_0^1)$, $n\in\mathbb{N}$, be defined by: $g_0(t)=U(t)$, $g_1(t)=U(t+r)$, ..., $g_n(t)=U(t+nr)$..., for $-r\leqslant t\leqslant r$. Then, $g_n(t)=\Phi^n(g_0(t)) \xrightarrow{n\to\infty} \hat{\psi}(t)$, a.e. in t, where $\hat{\psi}(t)=\psi(t)$ for $-r\leqslant t\leqslant 0$ and $\hat{\psi}(t)=\psi(t-r)$ for $0\leqslant t\leqslant r$. Let us look for a function in $L_1([-r, r])$ that dominates $\|g_n(t)\|_{H_0^1}^2$.

By Proposition 3.1, we have

$$\frac{1-k}{2}\|p\|_{H_0^1}^2\leqslant V(p)+\frac{KM^2}{2}$$

for all p in H_0^1, thus

$$\frac{1-k}{2}\|g_n(t)\|_{H_0^1}^2\leqslant V(\Phi^n(g_0(t)))+\frac{KM^2}{2}\leqslant V(g_0(t))+\frac{KM^2}{2}$$

and the function of $t\in[-r, r]$ in the last member is in $L_1([-r, r])$.

Then, by Lebesgue-Dominated Convergence Theorem, we have that $g_n \xrightarrow{n\to\infty} \hat{\psi}$ in $L_2([-r, r], H_0^1)$. Thus, given $\delta > 0$, $\exists n_\delta \in \mathbb{N}$ such that $n \geqslant n_\delta \Rightarrow \|g_n - \hat{\psi}\|_2^{[-r,r]} < \delta$ and for $t \geqslant n_\delta r$, let $n \geqslant n_\delta$ be such that $nr \leqslant t < (n+1)\,r$. Then $\|S(t)\,\varphi - S(t)\,\psi\|_2^{\#} \leqslant \|g_n - \hat{\psi}\|_2^{[-r,r]} < \delta$.

This means that $S(t)\,\varphi \in \mathscr{B}_\delta(\gamma(\psi))$ for $t \geqslant n_\delta r$ where $\mathscr{B}_\delta(\gamma(\psi))$ is the set of elements of $L_2^{\#}$ with distance from $\gamma(\psi)$ less than δ. ∎

Corollary 3.1. *$\gamma^+(\varphi)$, the positive orbit of φ, is precompact, $\forall \varphi \in L_2^{\#}$.*

Proof. For a given φ, let ψ be as in theorem. Since $\gamma(\psi) = S([0, r])\,\psi$ is compact and attracts φ, we see that the Kuratowski measure of non-compactness of $\gamma^+(\varphi)$ is zero. ∎

Corollary 3.2. *$\mathscr{E}$ attracts points of $L_2^{\#}$.*

Proof. For $\varphi \in L_2^{\#}$ and $\delta > 0$, take $\psi \in \mathscr{E}$ and $n_\delta \in \mathbb{N}$ as in theorem. For $t \geqslant n_\delta r$ we have: $S(t)\,\varphi \in \mathscr{B}_\delta(\gamma(\psi)) \subset \mathscr{B}_\delta(\mathscr{E})$. ∎

Definition 3.2. The set $\mathscr{A} \stackrel{\text{def}}{=} \{\varphi \in L_2^{\#} \mid \varphi(\theta) \in A$ a.e. in $\theta \in [-r, 0]\}$.

Since A is bounded (and even compact) in H_0^1, we note that $\mathscr{A} \subset L_\infty^{\#}$ and is bounded (even with the $\|\ \|_\infty^{\#}$-norm). $\mathscr{A}$ is closed but not compact unless $E = \{0\}$.

Proposition 3.5. *We have $\mathscr{A} = W^u(\mathscr{E})$. Moreover $\mathscr{A} = \bigcup_{\psi \in \mathscr{E}} W^u(\gamma(\psi))$, the union of the unstable manifolds of the periodic orbits of $\mathscr{E}$.*

Proof. For $\varphi \in \mathscr{A}$ we can extend $S(\cdot)\,\varphi$ for $t \leqslant 0$ since Φ^{-1} is continuous in $A \subset H_0^1$ (see [1]). Since $\varphi(\theta) \in A = \bigcup_{i=0}^m W^u(v_i)$, $v_i \in E$ (see Remark 3.4), we have that $\exists \psi(\theta) \in E$ such that $\varphi(\theta) \in W^u(\psi(\theta))$. Then $\lim_{n\to\infty} \Phi^{-n}(\varphi(\theta)) = \psi(\theta)$ and $\psi \in \mathscr{E}$. As in Theorem 3.1, we may show that the α-limit of φ, $\alpha(\varphi)$, is the orbit of ψ: $\gamma(\psi)$. To prove this we define $g_0, g_1, ..., g_n...$ in $L_2([-2r, 0], H_0^1)$ by $g_n(t) = U(t - nr)$, $n \in \mathbb{N}$, and we have $g_n(t) = \Phi^{-n}(g_0(t)) \xrightarrow{n\to\infty} \hat{\psi}(t)$, a.e. in t.

Thus

$$\frac{1-k}{2}\,\|g_n(t)\|_{H_0^1}^2 \leqslant V(\Phi^{-n}(g_0(t))) + \frac{KM^2}{2} \leqslant V(\hat{\psi}(t)) + \frac{KM^2}{2}$$

and this last function of t is in $L_1([-2r, 0])$. ($V(\Phi^{-n}(g_0(t))) \xrightarrow{n\to\infty} V(\hat{\psi}(t))$ *increasingly*, since V is nonincreasing in the orbits of Φ).

Thus $\varphi \in W^u(\gamma(\psi))$. Hence $\mathscr{A} \subset \bigcup_{\psi \in \mathscr{E}} W^u(\gamma(\psi)) \subset W^u(\mathscr{E})$.

On the other hand, if $\varphi \in W^u(\mathscr{E})$, i.e., if $\alpha(\varphi) \subset \mathscr{E}$, we have by the definition of α-limit, that there exist $\psi \in \mathscr{E}$ and a sequence $\{t_n\}_{n\in\mathbb{N}}$, $t_n \xrightarrow{n\to\infty} \infty$, with $S(-t_n)\varphi \xrightarrow{n\to\infty} \psi$, in $L_2^\#$. Let l_n be in $[0, r[$ such that $-t_n + l_n = -k_n r$ with $k_n \in \mathbb{N}$ for all $n \in \mathbb{N}$. Taking a subsequence, if necessary, we have that $l_n \xrightarrow{n\to\infty} l \in [0, r]$. By Theorem 2.1 we know that $S\colon [0, r] \times L_2^\# \to L_2^\#$ is continuous. Then $S(-k_n r)\varphi = S(l_n)[S(-t_n)\varphi] \xrightarrow{n\to\infty} S(l)[\psi]$, in $L_2^\#$, which implies that, for a subsequence, we have the pointwise convergence a.e. in θ, i.e., $\Phi^{-k_{n_i}}(\varphi(\theta)) \xrightarrow{k_{n_i}\to\infty} \tilde{\psi}(\theta)$ where $\tilde{\psi} = S(l)\psi \in \mathscr{E}$. Hence $\varphi(\theta) \in W^u(\tilde{\psi}(\theta)) \subset A$ a.e. in θ. Therefore $\varphi \in \mathscr{A}$. ▮

PROPOSITION 3.6. *$\mathscr{A}$ attracts bounded sets of $L_\infty^\#$ in the $\| \ \|_\infty^\#$-norm (and hence in $\| \ \|_2^\#$ too).*

Proof. Let $\mathscr{B} \subset L_\infty^\#$ be bounded in the $\| \ \|_\infty^\#$-norm, that is, $\exists \bar{M} > 0$ such that $\varphi(\theta) \in B_{\bar{M}}(0)$ (ball of H_0^1) a.e. in θ. Since A attracts bounded sets of H_0^1 (see Remark 3.3), we have that given $\delta > 0$, $\exists n_\delta \in \mathbb{N}$ such that $n > n_\delta \Rightarrow \Phi^n(\varphi(\theta)) \in B_\delta(A)$ (the points of H_0^1 whose distance from A is less than δ). Then, for $t > n_\delta r$ we have $(S(t)\varphi)(\theta) \in B_\delta(A)$. For each $\varphi \in \mathscr{B}$ and $t > n_\delta r$, we can choose a simple function η near $S(t)\varphi$ in $L_\infty^\#$, with values in $B_{2\delta}(A)$. Then we can choose a simple function ψ with values in A, hence $\psi \in \mathscr{A}$, such that $\|\eta(\theta) - \psi(\theta)\|_{H_0^1} \leqslant 2\delta$ a.e. in θ. Hence we have $S(t)\varphi$ near ψ in the $\| \ \|_\infty^\#$-norm because $\|S(t)\varphi(\theta) - \psi(\theta)\|_{H_0^1} \leqslant \|S(t)\varphi(\theta) - \eta(\theta)\|_{H_0^1} + \|\eta(\theta) - \psi(\theta)\|_{H_0^1} < \delta + 2\delta = 3\delta$, a.e. in θ, i.e., $S(t)\varphi \in \mathscr{B}_{3\delta}^\infty(\mathscr{A}) \subset \mathscr{B}_{3\delta\sqrt{r}}(\mathscr{A})$, $\forall \varphi \in \mathscr{B}$, for $t > n_\delta r$, where $\mathscr{B}_{3\delta}^\infty(\mathscr{A})$ means the elements of $L_\infty^\#$ whose distance in the $\| \ \|_\infty^\#$-norm from $\mathscr{A}$ is less than 3δ; and $\mathscr{B}_{3\delta\sqrt{r}}(\mathscr{A})$ means the elements of $L_2^\#$ whose distance in $\| \ \|_2^\#$-norm from $\mathscr{A}$ is less than $3\delta\sqrt{r}$. ▮

THEOREM 3.2. *$\mathscr{A}$ is a stable set, that is, given $\zeta > 0$, $\exists \delta > 0$ and $T > 0$ such that*

$$\varphi \in \mathscr{B}_\delta(\mathscr{A}) \Rightarrow S(t)\varphi \in \mathscr{B}_\zeta(\mathscr{A}) \qquad \textit{for} \quad t \geqslant T.$$

Proof. Given $\delta > 0$, φ is in $\mathscr{B}_\delta(\mathscr{A})$ iff $\exists \xi \in \mathscr{A}$ such that $\|\xi - \varphi\|_2^\# < \delta$. Given a number $R > 0$ let us call $G_1 = G_1(\varphi) = \{\theta \in [-r, 0] / \|\varphi(\theta)\|_{H_0^1} \geqslant R\}$, $G_2 = [-r, 0] \backslash G_1$, $\varphi_1 = \varphi \mathscr{X}_{G_1}$ and $\varphi_2 = \varphi \mathscr{X}_{G_2}$. Then $\varphi = \varphi_1 + \varphi_2$. Let μ be the Lebesgue-measure in $\mathbb{R}$.

For φ in $\mathscr{B}_\delta(\mathscr{A})$ we have

$$\mu(G_1) \leqslant \eta \stackrel{\text{def}}{=} \sup_{\bar{\varphi} \in \mathscr{B}_\delta(\mathscr{A})} \mu G_1(\bar{\varphi}) \leqslant \left(\frac{\delta}{R - \tilde{m}}\right)^2,$$

when $R > \tilde{m}$, where $\tilde{m} = \sup\{\|v\|_{H_0^1} \mid v \in A\}$. Take ψ in $\mathscr{E}$ such that $\lim_{n\to\infty} \Phi^n(\varphi_1(\theta)) = \psi(\theta)$ a.e. in θ, as in Theorem 3.1, and put $l = $ *diam* $A = \sup\{\|v - \omega\|_{H_0^1} \mid v, \omega \in A\}$.

We have: $\|\varphi_1(\theta)-\psi(\theta)\|_{H_0^1} \leqslant \|\varphi_1(\theta)-\xi(\theta)\|_{H_0^1} + \|\xi(\theta)-\psi(\theta)\|_{H_0^1} \leqslant \|\varphi_1(\theta)-\xi(\theta)\|_{H_0^1} + l$ a.e. in θ. Then:

$$\begin{aligned}\|\varphi_1-\psi\|_2^{\#} &= \|\varphi_1-\psi\|_2^{G_1}\\ &\leqslant \|\varphi_1-\xi\|_2^{G_1} + l\sqrt{\mu(G_1)}\\ &\leqslant \|\varphi-\xi\|_2^{\#} + l\sqrt{\eta}\\ &< \delta + l\sqrt{\eta}\end{aligned}$$

Then, for a given $\delta'>0$, if we take $\delta<\delta'/2$ and R big enough such that $\eta<\delta'^2/4l^2$ we get $\|\varphi_1-\psi\|_2^{\#}<\delta'$.

Now, for p in H_0^1, from Proposition 3.1, we get

$$\begin{aligned}&\frac{1-k}{2}\|\Phi^n p\|_{H_0^1}^2 - \frac{KM^2}{2}\\ &\qquad \leqslant V(\Phi^n(p)) \leqslant V(p) \leqslant \frac{(1+K)}{2}\|p\|_{H_0^1}^2 \qquad \forall n\in\mathbb{N},\end{aligned}$$

thus

$$\|\Phi^n p\|_{H_0^1}^2 \leqslant \frac{1+K}{1-k}\|p\|_{H_0^1}^2 + \frac{KM^2}{1-k} = c\,\|p\|_{H_0^1}^2 + d.$$

Let v be in E such that $\lim_{n\to\infty}\Phi^n p = v$, as in Remark 3.4, and put $m=\sup\{\|v\|_{H_0^1} \mid v\in E\}$. Then

$$\begin{aligned}\|\Phi^n(p)-v\|_{H_0^1} &\leqslant \|\Phi^n(p)\|_{H_0^1} + \|v\|_{H_0^1}\\ &\leqslant \sqrt{c\,\|p\|_{H_0^1}^2 + d} + m\\ &\leqslant \sqrt{c(\|p-v\|_{H_0^1}+m)^2 + d} + m,\end{aligned}$$

and if $\|p-v\|_{H_0^1}>m$ and $\|p-v\|_{H_0^1}>\sqrt{d}$ we get

$$\|\Phi^n p - v\|_{H_0^1} \leqslant (\sqrt{4c+1}+1)\,\|p-v\|_{H_0^1} = b\,\|p-v\|_{H_0^1}.$$

For $t>0$, let $\bar{n}\in\mathbb{N}$ be such that $\bar{n}r\leqslant t<(\bar{n}+1)\,r$. Then

$$\begin{aligned}\|S(t)\,\varphi_1 - S(t)\,\psi\|_2^{\#} &\leqslant \|S((\bar{n}+1)\,r)\,\varphi_1-\psi\|_2^{\#} + \|S(\bar{n}r)\,\varphi_1-\psi\|_2^{\#}\\ &= \|\Phi^{\bar{n}+1}\circ\varphi_1-\psi\|_2^{\#} + \|\Phi^{\bar{n}}\circ\varphi_1-\psi\|_2^{\#}\end{aligned}$$

and

$$\|\Phi^n \circ \varphi_1 - \psi\|_2^{\#^2} = \int_{G_1} \|\Phi^n(\varphi_1(\theta)) - \psi(\theta)\|_{H_0^1}^2 \, d\theta$$

$$\leqslant \int_{G_1} b^2 \, \|\varphi_1(\theta) - \psi(\theta)\|_{H_0^1}^2 \, d\theta$$

$$= b^2 \, \|\varphi_1 - \psi\|_2^{\#^2} \leqslant b^2 \delta'^2 \qquad \forall n \in \mathbb{N},$$

i.e., $\|S(t)\,\varphi_1 - S(t)\,\psi\|_2^{\#} \leqslant b\delta' + b\delta' = 2b\delta'$, whenever we have $\|\varphi_1(\theta) - \psi(\theta)\|_{H_0^1} > \max\{m, \sqrt{d}\}$ a.e. in $\theta \in G_1$, but for this it is enough to take $R > \max\{m, \sqrt{d}\} + m$ since $\|\varphi_1(\theta) - \psi(\theta)\|_{H_0^1} \geqslant \|\varphi_1(\theta)\|_{H_0^1} - \|\psi(\theta)\|_{H_0^1} \geqslant R - m$.

Then, given $\zeta > 0$, we take $\delta' < \zeta/4b$ and take δ and R with the conditions above. So, for $\varphi \in \mathscr{B}_\delta(\mathscr{A})$ we get

$$\|S(t)\,\varphi_1 - S(t)\,\psi\|_2^{\#} \leqslant \frac{\zeta}{2}, \qquad \forall t > 0.$$

This means that $S(t)\,\varphi_1 \in \mathscr{B}_{\zeta/2}(S(t)\,\psi) \subset \mathscr{B}_{\zeta/2}(\gamma(\psi)) \subset \mathscr{B}_{\zeta/2}(\mathscr{E}) \subset \mathscr{B}_{\zeta/2}(\mathscr{A})$, $\forall t > 0$. Now, for φ_2, we have $\varphi_2 \in \mathscr{B}_R^\infty(0)$ (ball of $L_\infty^{\#}$), then, by Proposition 3.6, $\exists T > 0$ which depends only on R, such that for $t > T$ we have $S(t)\,\varphi_2 \in \mathscr{B}^\infty_{\zeta/2\sqrt{r}}(\mathscr{A}) \subset \mathscr{B}_{\zeta/2}(\mathscr{A})$. Therefore, $S(t)\,\varphi = S(t)\,\varphi_1 + S(t)\,\varphi_2 \in \mathscr{B}_\zeta(\mathscr{A})$ for $t > T$. ∎

COROLLARY 3.3. *$\mathscr{A}$ attracts compact sets of $L_2^{\#}$.*

Proof. Let $\mathscr{K}$ be a compact set of $L_2^{\#}$. Given $\zeta > 0$, let $\delta > 0$ and $T > 0$ be as in Theorem 3.2. Since $\mathscr{E}$ attracts points of $L_2^{\#}$, for each $\varphi \in \mathscr{K}$ we have that $\exists t_\varphi > 0 \mid S(t_\varphi)\,\varphi \in \mathscr{B}_\delta(\mathscr{E})$. Since $S(t_\varphi)$ is continuous in φ, there exists V_φ, a neighbourhood of φ, such that $S(t_\varphi)\,V_\varphi \subset \mathscr{B}_\delta(\mathscr{E}) \subset \mathscr{B}_\delta(\mathscr{A})$. $\{V_\varphi\}_{\varphi \in \mathscr{K}}$ is a covering of $\mathscr{K}$ and we have $V_{\varphi_1}, ..., V_{\varphi_m}$ a finite subcovering. For $t \geqslant t_{\varphi_i} + T$ we have $S(t)\,V_{\varphi_i} \subset \mathscr{B}_\zeta(\mathscr{A})$, by Theorem 3.2. Then, for $t \geqslant \max\{t_{\varphi_i}\} + T$ we have $S(t)\mathscr{K} \subset \mathscr{B}_\zeta(\mathscr{A})$. ∎

So, the system (5) is a compact-dissipative system in $L_2^{\#}$.

4. HYPERBOLICITY OF THE FIXED POINTS AND THE NON LOCAL CONJUGACY WITH THE LINEARIZED SYSTEM

Let us suppose, from now on, that we are in the hypothesis of Theorem 2.2.

In [2], for neutral equations in L_p-phase-spaces, which include our case, we obtain a Hartman–Grobman-like conjugation of the equation with its linearized equation around a hyperbolic fixed point, not in a neighbourhood of the fixed point, if $p \neq \infty$, but only in a small invariant set around the fixed point, formed by functions of L_∞ whose distance in $\| \ \|_\infty$-norm from the fixed point is small (see [2] Theorem 5.5).

We show here that, for equation (5) in $L_2^\#$, this can not be improved, that is, we have an example of dynamical system with hyperbolic fixed points which is not locally topologically equivalent to its linearized system around each fixed point.

A fixed point of system (5) is an element of $\mathscr{E}$ with the form $\varphi_v = v\mathscr{X}_{[-r,0]}$, i.e., $\varphi_v(\theta) = v$ a.e. in θ, for $v \in E$. By Theorem 2.2, the linearized of equation (5) around this fixed point is the autonomous linear equation

$$Z(t) = D\Phi(v)\, Z(t-r) \qquad \text{for} \quad t > 0. \tag{9}$$

Put $L = D\Phi(v) \in \mathscr{L}(H_0^1)$ which is a compact linear operator (see Remark 3.3).

Let $\{T^\circ(t)\}_{t \geqslant 0}$ be the flow of (9), that is, for $\Delta\varphi$ in $L_2^\#$, $T^\circ(t)\,\Delta\varphi = Z_t$, the solution of (9) with $Z_0 = \Delta\varphi$, $T^\circ(t) \in \mathscr{L}(L_2^\#)$ for $t \geqslant 0$. By Theorem 2.2 we have $T^\circ(t) = DS(t)\,\varphi_v$, $t > 0$, the Hadamard derivative of $S(t)$ at the fixed point φ_v.

In [4], a Daniel Henry's manuscript, we have some results for linear difference equations in Hilbert spaces, which include equation (9). These results are shown in [6] Chap. III Section 3 from Theorem 4 until Theorem 7 (see also [2] Remark 4.2).

Simplifying these results for equation (9) we obtain:

THEOREM 4.1. *For equation* (9) *we have*:

(i) *The infinitesimal generator of the semi-group* $\{T^\circ(t)\}_{t \geqslant 0}$ *is given by* $G^\circ\varphi = \dot{\varphi}$, *the derivate of* φ, *with domain* $\mathscr{D}(G^\circ) = \{\varphi \in L_2^\# \mid \varphi$ *is absolutely continuous*, $\dot{\varphi} \in L_2^\#$ *and* $\varphi(0) = L(\varphi(-r))\}$.

(ii) *The point spectrum* (*the eigenvalues*) *of* G° *is* $P_\sigma(G^\circ) = \{\mu \in \mathbb{C} \mid 1 \in P_\sigma(Le^{-\mu r})\}$.

(iii) *For each* $t \geqslant 0$, *the spectrum of* $T^\circ(t)$, $\sigma(T^\circ(t))$, *is characterized by* $\sigma(T^\circ(t)) \backslash \{0\} = \overline{e^{tP_\sigma(G^\circ)}} \backslash \{0\}$ *where for* $B \subset \mathbb{C}$, $\bar{B}$ *means the closure of* B *in* $\mathbb{C}$.

Remark 4.1. We know that $P_\sigma(L) = \{\lambda_0, \lambda_1, ..., \lambda_n ...\}$ with $\lambda_0 > \lambda_1 > \cdots > \lambda_n \to 0$, all real and simple with $\lambda_j \neq 1$ for $j \in \mathbb{N}$ (see Remark 3.3). From (ii) of the above theorem, we conclude that $\mu \in P_\sigma(G^\circ) \Leftrightarrow e^{\mu r} \in P_\sigma(L) \Leftrightarrow e^{\mu r} = \lambda_j$, $j \in \mathbb{N} \Leftrightarrow \mu = \ln \lambda_j + (i\, 2k\pi)/r$, $j \in \mathbb{N}$, $k \in \mathbb{Z}$.

Hence, from (iii) of the theorem, we get: $\sigma(T^\circ(t)) = \overline{\{\lambda_j^{t/r} \cdot e^{i(t/r)\, 2k\pi} \in \mathbb{C} \mid j \in \mathbb{N}, k \in \mathbb{Z}\}}$, for $t \geqslant 0$. For $t > 0$, this set is contained in the circumferences of radius $\lambda_j^{t/r} \neq 1$, for $j \in \mathbb{N}$, then it does not intercept the unitary circumference of $\mathbb{C}$. Hence $T^\circ(t)$ is hyperbolic for $t > 0$, i.e., φ_v is a hyperbolic fixed point of system (5).

Equation (9), the linearized of equation (5) around the hyperbolic fixed point φ_v, satisfies the condition given in [2]-Proposition 5.1, and has the canonical linear hyperbolic behavior in $L_2^\#$, according to this proposition ($L_2^\# = L_2^s \oplus L_2^u$ the direct sum of the stable and unstable subspaces for the fixed point 0 which is the unique non-wandering point of the system, etc.). Then, there is no local topological equivalence of system (5) around φ_v, and system (9) around 0, since at any neighborhood of φ_v in $L_2^\#$ we find periodic orbits of period r of the system (5) totally inside the neighborhood (for example, for each $\delta > 0$, the distance in $L_2^\#$ from $v\mathscr{X}_{[-r,\,-\delta]}$ (as well from each function of its r-periodic orbit) to φ_v is $\|v\|_{H_0^1} \sqrt{\delta}$).

We can not speak about transversal section of a periodic orbit of $\mathscr{E}$ since we don't have a tangent direction to the orbit at some point because $S(t)\,\psi$ is not differentiable at t for ψ in $\mathscr{E}$. So, we can not define Poincaré's transformation, and hyperbolicity of the orbit in the canonical way, but we show that each $\psi \in \mathscr{E}$ is a hyperbolic fixed point of $S(r)$. This fact gives us the possibility of defining hyperbolic periodic orbit for this kind of dynamical system, since it suffices, for a dynamical system coming from a vector field, verify the hyperbolicity of $S(r)$ (say, the flow at time r which is the period of the orbit) just for transversal sections of the orbit at each point ($S(r)$ always has the eigenvalue 1 in the tangent direction to the orbit).

THEOREM 4.2. *$\psi \in \mathscr{E} \Rightarrow \psi$ is a hyperbolic fixed point of $S(r)$.*

Proof. Let us write $\psi = \sum_{i=1}^m v_i \mathscr{X}_{G_i}$, $v_i \in E \subset H_0^1$ and $\{G_i\}_{i=1,2,\ldots,m}$ is a disjoint measurable covering of $[-r, 0]$. For each $\theta \in [-r, 0]$, let $i_\theta \in \{1, 2, \ldots, m\}$ be such that $\psi(\theta) = v_{i_\theta} \in E$. For $t \geqslant 0$ put $i_t = i_{t-r}$. Let V be the solution of (5) through ψ, that is, $V_t = S(t)\,\psi$, $V(t) = \Phi(V(t-r)) = V(t-r) = \psi(t-r) = v_{i_t}$ for $t > 0$. For the linearization around the solution V we have, following Theorem 2.2, the linear non-autonomous equation:

$$Z(t) = D\Phi(v(t-r)) \cdot Z(t-r), \qquad t > 0. \tag{10}$$

Put $L_i = D\Phi(v_i) \in \mathscr{L}(H_0^1)$, $i = 1, 2, \ldots, m$. Then we can write equation (10) as $Z(t) = L_{i_t}(Z(t-r))$.

We know from Theorem 2.2 that, for $\Delta\psi \in L_2^\#$, we have $[DS(t)\,\psi]\,\Delta\psi = Z_t$, the solution of (10) with $Z_0 = \Delta\psi$. Then $[DS(r)\,\psi]\,\Delta\psi = Z_r$ where $Z_r(\theta) = Z(r+\theta) = L_{i_\theta}(\Delta\psi(\theta))$, $\theta \in [-r, 0]$. The covering $\{G_i\}_{i=1,\ldots,m}$ of $[-r, 0]$ allows us to decompose $L_2^\#$ as a direct sum of closed subspaces $\wedge_i = \{\varphi \in L_2^\# \mid \varphi(\theta) = 0 \text{ a.e. in } \theta \in [-r, 0] \backslash G_i\}$, $i = 1, 2, \ldots, m$. Then, for $\varphi \in L_2^\#$ we have the unique decomposition $\varphi = \varphi_1 + \varphi_2 + \cdots \varphi_m$ with $\varphi_i \in \wedge_i$, $i = 1, \ldots, m$.

Let $\{T_i^\circ(t)\}_{t \geqslant 0}$ be the flow of the autonomous linear equation (9) when we take $v = v_i$ and hence $L = L_i$ for $i = 1, 2, ..., m$.

We note that the subspaces $\wedge_i$ are invariant under $S(r)$, $DS(r)\,\psi$ and $T_i^\circ(r)$ for $i = 1, 2, ..., m$. Moreover we have $[DS(r)\,\psi]\,\Delta\psi = L_1 \circ \Delta\psi_1 + L_2 \circ \Delta\psi_2 + \cdots L_m \circ \Delta\psi_m = T_1^\circ(r)\,\Delta\psi_1 + T_2^\circ(r)\,\Delta\psi_2 + \cdots T_m^\circ(r)\,\Delta\psi_m$.

Let us prove that $\sigma(DS(r)\,\psi) \subset \bigcup_{i=1}^m \sigma(T_i^\circ(r))$ and this will be sufficient to show that $DS(r)\,\psi$ is hyperbolic since each $T_i^\circ(r)$ is hyperbolic (see Remark 4.1).

To show that the spectrum of $DS(r)\,\psi$ is contained in a union of spectra, let us pass to the complement in $\mathbb{C}$ of the spectrum of an operator T which is the resolvent-set, $\rho(T)$, of the operator. We will show that $\rho(DS(r)\,\psi) \supset \bigcap_{i=1}^m \rho(T_i^\circ(r))$.

Suppose $\lambda \in \bigcap_{i=1}^m \rho(T_i^\circ(r))$. This means that for each $\varphi_i \in L_2^\#$, there exists a unique $\xi_i \in L_2^\#$ such that $\lambda\xi_i - T_i^\circ(r)\,\xi_i = \varphi_i$ for $i = 1, 2, ..., m$. Given $\varphi \in L_2^\#$, we have $\varphi = \varphi_1 + \cdots + \varphi_m$, $\varphi_i \in \wedge_i$, $i = 1, ..., m$, and for $\varphi_i \in \wedge_i \subset L_2^\#$, take ξ_i as above, $i = 1, 2, ..., m$. Let us show that $\xi_i \in \wedge_i$, $i = 1, ..., m$. For θ in $[-r, 0] \backslash G_i$, we have a.e. $\lambda\xi_i(\theta) - [T_i^\circ(r)\,\xi_i](\theta) = \varphi_i(\theta) = 0$, then $[T_i^0(r)\,\xi_i](\theta) = L_i(\xi_i(\theta)) = \lambda\xi_i(\theta)$. If $\xi_i(\theta) \neq 0$ we would have $\lambda \in P_\sigma(L_i)$ but $P_\sigma(L_i) \subset \sigma(T_i^0(r))$ (see $\sigma(T^0(t))$ in Remark 4.1 and take $t = r$). Since $\lambda \in \rho(T_i^0(r))$ we have $\xi_i(\theta) = 0$ a.e. outside of G_i. Then $\xi_i \in \wedge_i$, $i = 1, 2, ..., m$. Write $\xi = \xi_1 + \xi_2 + \cdots + \xi_m$. We have: $(\lambda\xi_1 - T_1^0(r)\,\xi_1) + (\lambda\xi_2 + T_2^0(r)\,\xi_2) + \cdots + (\lambda\xi_m + T_m^0(r)\,\xi_m) = \varphi_1 + \varphi_2 + \cdots \varphi_m = \varphi$, that is, $\lambda(\xi_1 + \cdots + \xi_m) - (T_1^0(r)\,\xi_1 + T_2^0(r)\,\xi_2 + \cdots T_m^0(r)\xi_m) = \lambda\xi - [DS(r)\,\psi]\,\xi = \varphi$. This means that for each φ in $L_2^\#$ we found a unique $\xi \in L_2^\#$ such that $(\lambda I - DS(r)\,\psi)\,\xi = \varphi$. By the Open Mapping Theorem $(\lambda I - DS(r)\,\psi)^{-1}$ is continuous, i.e., $\lambda \in \rho(DS(r)\,\psi)$. ∎

Remark 4.2. If the period of the orbit of $\psi \in \mathscr{E}$ is r/k with $k \in \mathbb{N}^*$, using the same arguments of Theorem 4.2, we can show that ψ is a hyperbolic fixed point of $S(r/k)$. Hence, we can say that the orbits of $\mathscr{E}$ are all hyperbolic.

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